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JOHN BIRD

SEVENTH EDITION

ROUTLEDGE

Higher Engineering Mathematics

Why is knowledge of mathematics important in engineering?

A career in any engineering or scientific field will require both basic and advanced mathematics. Without mathematics to determine principles, calculate dimensions and limits, explore variations, prove concepts, and so on, there would be no mobile telephones, televisions, stereo systems, video games, microwave ovens, computers, or virtually anything electronic. There would be no bridges, tunnels, roads, skyscrapers, automobiles, ships, planes, rockets or most things mechanical. There would be no metals beyond the common ones, such as iron and copper, no plastics, no synthetics. In fact, society would most certainly be less advanced without the use of mathematics throughout the centuries and into the future.

Electrical engineers require mathematics to design, develop, test, or supervise the manufacturing and installation of electrical equipment, components, or systems for commercial, industrial, military, or scientific use.

Mechanical engineers require mathematics to perform engineering duties in planning and designing tools, engines, machines, and other mechanically functioning equipment; they oversee installation, operation, maintenance, and repair of such equipment as centralised heat, gas, water, and steam systems.

Aerospace engineers require mathematics to perform a variety of engineering work in designing, constructing, and testing aircraft, missiles, and spacecraft; they conduct basic and applied research to evaluate adaptability of materials and equipment to aircraft design and manufacture and recommend improvements in testing equipment and techniques.

Nuclear engineers require mathematics to conduct research on nuclear engineering problems or apply principles and theory of nuclear science to problems concerned with release, control, and utilisation of nuclear energy and nuclear waste disposal.

Petroleum engineers require mathematics to devise methods to improve oil and gas well production and determine the need for new or modified tool designs; they oversee drilling and offer technical advice to achieve economical and satisfactory progress.

Industrial engineers require mathematics to design, develop, test, and evaluate integrated systems for managing industrial production processes, including human work factors, quality control, inventory control, logistics and material flow, cost analysis, and production co-ordination.

Environmental engineers require mathematics to design, plan, or perform engineering duties in the prevention, control, and remediation of environmental health hazards, using various engineering disciplines; their work may include waste treatment, site remediation, or pollution control technology.

Civil engineers require mathematics in all levels in civil engineering – structural engineering, hydraulics and geotechnical engineering are all fields that employ mathematical tools such as differential equations, tensor analysis, field theory, numerical methods and operations research.

Knowledge of mathematics is therefore needed by each of the engineering disciplines listed above.

It is intended that this text – *Higher Engineering Mathematics* – will provide a step-by-step approach to learning fundamental mathematics needed for your engineering studies.

To Sue

Higher Engineering Mathematics

Seventh Edition

John Bird, BSc (Hons), CMath, CEng, CSci, FITE, FIMA, FCollT

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Contents

Preface

This **seventh edition of *Higher Engineering Mathematics*** covers essential mathematical material suitable for students studying **Degrees, Foundation Degrees, and Higher National Certificate and Diploma courses in Engineering disciplines.**

The text has been conveniently divided into the following **12 convenient categories**: number and algebra, geometry and trigonometry, graphs, complex numbers, matrices and determinants, vector geometry, differential calculus, integral calculus, differential equations, statistics and probability, Laplace transforms and Fourier series.

Increasingly, **difficulty in understanding algebra** is proving a problem for many students as they commence studying engineering courses. Inevitably there are a lot of formulae and calculations involved with engineering studies that require a sound grasp of algebra. On the website, available to all, is a document which offers **a quick revision of the main areas of algebra** essential for further study, i.e. basic algebra, simple equations, transposition of formulae, simultaneous equations and quadratic equations.

For this edition, **new material** has been added on loci, eigenvalues and eigenvectors, points of inflexion, double and triple integrals, permutations and combinations and Laplace transforms of the Heaviside function, together with material that was previously on the website, that is, inequalities, Boolean algebra and logic circuits, sampling and estimation theories, significance testing, and Chi square and distribution-free tests.

The **primary aim of the material in this text** is to provide the fundamental analytical and underpinning knowledge and techniques needed to successfully complete scientific and engineering principles modules of Degree, Foundation Degree and Higher National Engineering programmes. The material has been designed to enable students to use techniques learned for the analysis, modelling and solution of realistic engineering problems at Degree and Higher National level. It also aims to provide some of the more advanced knowledge required for those wishing to pursue careers in mechanical engineering, aeronautical engineering, electrical and electronic engineering, communications engineering, systems engineering and all variants of control engineering.

In ***Higher Engineering Mathematics 7th Edition***, theory is introduced in each chapter by a full outline of essential definitions, formulae, laws, procedures, etc; **problem solving** is extensively used to establish and exemplify the theory. It is intended that readers will gain real understanding through seeing problems solved and then through solving similar problems themselves.

Access to **software packages** such as Maple, Mathematica and Derive, or a graphics calculator, will enhance understanding of some of the topics in this text.

Each topic considered in the text is presented in a way that assumes in the reader only knowledge attained in BTEC National Certificate/Diploma, or similar, in an Engineering discipline.

Higher Engineering Mathematics 7th Edition* provides a follow-up to *Engineering Mathematics 7th Edition.

This textbook contains some **1020 worked problems**, followed by over **1900 further problems (with answers)**, arranged within **269 Practice Exercises**. Some **512 line diagrams** further enhance understanding.

Worked solutions to all 1900 of the further problems has been prepared and can be **accessed free by students and staff via the website** (see page xiv).

At the end of the text, a list of **Essential Formulae** is included for convenience of reference.

At intervals throughout the text are some **20 Revision Tests** to check understanding. For example, Revision Test 1 covers the material in Chapters 1 to 5, Revision Test 2 covers the material in Chapters 6 to 8, Revision Test 3 covers the material in Chapters 9 to 11, and so on. An **Instructor's Manual**, containing full solutions to the Revision Tests, is available free to lecturers/instructors via the website (see page xiv).

'Learning by example' is at the heart of *Higher Engineering Mathematics 7th Edition*.

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Free Web downloads

The following support material is available from http://www.routledge.com/bird/

For Students:

1. Full solutions to all 1900 further questions contained in the 269 Practice Exercises
2. Revision of some important algebra topics
3. A list of Essential Formulae
4. Information on 31 Mathematicians/Engineers mentioned in the text

For Lecturers/Instructors:

1. Full solutions to all 1900 further questions contained in the 269 Practice Exercises
2. Revision of some important algebra topics
3. Full solutions and marking scheme for each of the 20 Revision Tests; also, each test may be downloaded for distribution to students. In addition, solutions to the Revision Test given in the Revision of Algebra Topics is also included
4. A list of Essential Formulae
5. Information on 31 Mathematicians/Engineers mentioned in the text
6. All 512 illustrations used in the text may be downloaded for use in PowerPoint presentations

Syllabus guidance

This textbook is written for **undergraduate Engineering Degree and Foundation Degree courses**; however, it is also most appropriate for **BTEC levels 4 and 5 HNC/D studies in engineering** and three syllabuses are covered. The appropriate chapters for these three syllabuses are shown in the table below.

Chapter		**Analytical Methods for Engineers**	**Further Analytical Methods for Engineers**	**Advanced Mathematics for Engineering**
1.	Algebra	×		
2.	Partial fractions	×		
3.	Logarithms	×		
4.	Exponential functions	×		
5.	Inequalities			
6.	Arithmetic and geometric progressions	×		
7.	The binomial series	×		
8.	Maclaurin's series	×		
9.	Solving equations by iterative methods		×	
10.	Binary, octal and hexadecimal numbers		×	
11.	Boolean algebra and logic circuits		×	
12.	Introduction to trigonometry	×		
13.	Cartesian and polar co-ordinates	×		
14.	The circle and its properties	×		
15.	Trigonometric waveforms	×		
16.	Hyperbolic functions	×		
17.	Trigonometric identities and equations	×		
18.	The relationship between trigonometric and hyperbolic functions	×		
19.	Compound angles	×		
20.	Functions and their curves		×	
21.	Irregular areas, volumes and mean values of waveforms		×	
22.	Complex numbers		×	
23.	De Moivre's theorem		×	
24.	The theory of matrices and determinants		×	
25.	Applications of matrices and determinants		×	

(*Continued*)

Chapter		**Analytical Methods for Engineers**	**Further Analytical Methods for Engineers**	**Advanced Mathematics for Engineering**
26.	Vectors		×	
27.	Methods of adding alternating waveforms		×	
28.	Scalar and vector products		×	
29.	Methods of differentiation	×		
30.	Some applications of differentiation	×		
31.	Differentiation of parametric equations			
32.	Differentiation of implicit functions	×		
33.	Logarithmic differentiation	×		
34.	Differentiation of hyperbolic functions	×		
35.	Differentiation of inverse trigonometric and hyperbolic functions	×		
36.	Partial differentiation			×
37.	Total differential, rates of change and small changes			×
38.	Maxima, minima and saddle points for functions of two variables			×
39.	Standard integration	×		
40.	Some applications of integration	×		
41.	Integration using algebraic substitutions	×		
42.	Integration using trigonometric and hyperbolic substitutions	×		
43.	Integration using partial fractions	×		
44.	The $t = \tan\theta/2$ substitution			
45.	Integration by parts	×		
46.	Reduction formulae	×		
47.	Double and triple integrals			
48.	Numerical integration		×	
49.	Solution of first-order differential equations by separation of variables		×	
50.	Homogeneous first-order differential equations			
51.	Linear first-order differential equations		×	
52.	Numerical methods for first-order differential equations		×	×
53.	Second-order differential equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$		×	

(*Continued*)

Chapter		**Analytical Methods for Engineers**	**Further Analytical Methods for Engineers**	**Advanced Mathematics for Engineering**
54.	Second-order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$		×	
55.	Power series methods of solving ordinary differential equations			×
56.	An introduction to partial differential equations			×
57.	Presentation of statistical data	×		
58.	Mean, median, mode and standard deviation	×		
59.	Probability	×		
60.	The binomial and Poisson distributions	×		
61.	The normal distribution	×		
62.	Linear correlation	×		
63.	Linear regression	×		
64.	Sampling and estimation theories	×		
65.	Significance testing	×		
66.	Chi-square and distribution-free tests	×		
67.	Introduction to Laplace transforms			×
68.	Properties of Laplace transforms			×
69.	Inverse Laplace transforms			×
70.	The Laplace transform of the Heaviside function			
71.	Solution of differential equations using Laplace transforms			
72.	The solution of simultaneous differential equations using Laplace transforms			×
73.	Fourier series for periodic functions of period 2π			×
74.	Fourier series for non-periodic functions over range 2π			×
75.	Even and odd functions and half-range Fourier series			×
76.	Fourier series over any range			×
77.	A numerical method of harmonic analysis			×
78.	The complex or exponential form of a Fourier series			×

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Section A

Number and algebra

Chapter 1

Algebra

Why it is important to understand: **Algebra, polynomial division and the factor and remainder theorems**

It is probably true to say that there is no branch of engineering, physics, economics, chemistry or computer science which does not require the understanding of the basic laws of algebra, the laws of indices, the manipulation of brackets, the ability to factorise and the laws of precedence. This then leads to the ability to solve simple, simultaneous and quadratic equations which occur so often. The study of algebra also revolves around using and manipulating polynomials. Polynomials are used in engineering, computer programming, software engineering, in management, and in business. Mathematicians, statisticians and engineers of all sciences employ the use of polynomials to solve problems; among them are aerospace engineers, chemical engineers, civil engineers, electrical engineers, environmental engineers, industrial engineers, materials engineers, mechanical engineers and nuclear engineers. The factor and remainder theorems are also employed in engineering software and electronic mathematical applications, through which polynomials of higher degrees and longer arithmetic structures are divided without any complexity. The study of algebra, equations, polynomial division and the factor and remainder theorems is therefore of some considerable importance in engineering.

At the end of this chapter, you should be able to:

- understand and apply the laws of indices
- understand brackets, factorisation and precedence
- transpose formulae and solve simple, simultaneous and quadratic equations
- divide algebraic expressions using polynomial division
- factorise expressions using the factor theorem
- use the remainder theorem to factorise algebraic expressions

1.1 Introduction

In this chapter, polynomial division and the factor and remainder theorems are explained (in Sections 1.4 to 1.6). However, before this, some essential algebra revision on basic laws and equations is included.
For further algebra revision, go to the website:
www.routledge.com/cw/bird

1.2 Revision of basic laws

(a) Basic operations and laws of indices

The **laws of indices** are:

(i) $a^m \times a^n = a^{m+n}$ (ii) $\dfrac{a^m}{a^n} = a^{m-n}$

(iii) $(a^m)^n = a^{m\times n}$ (iv) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

(v) $a^{-n} = \dfrac{1}{a^n}$ (vi) $a^0 = 1$

Problem 1. Evaluate $4a^2bc^3 - 2ac$ when $a = 2$, $b = \frac{1}{2}$ and $c = 1\frac{1}{2}$

$$4a^2bc^3 - 2ac = 4(2)^2\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)^3 - 2(2)\left(\frac{3}{2}\right)$$

$$= \frac{4 \times 2 \times 2 \times 3 \times 3 \times 3}{2 \times 2 \times 2 \times 2} - \frac{12}{2}$$

$$= 27 - 6 = \mathbf{21}$$

Problem 2. Multiply $3x + 2y$ by $x - y$

$$\begin{array}{ll} & 3x + 2y \\ & x - y \\ \hline \text{Multiply by } x \rightarrow & 3x^2 + 2xy \\ \text{Multiply by } -y \rightarrow & \quad -3xy - 2y^2 \\ \hline \text{Adding gives:} & \mathbf{3x^2 - \ xy - 2y^2} \end{array}$$

Alternatively,

$$(3x + 2y)(x - y) = 3x^2 - 3xy + 2xy - 2y^2$$

$$= \mathbf{3x^2 - xy - 2y^2}$$

Problem 3. Simplify $\dfrac{a^3b^2c^4}{abc^{-2}}$ and evaluate when $a = 3$, $b = \frac{1}{8}$ and $c = 2$

$$\frac{a^3b^2c^4}{abc^{-2}} = a^{3-1}b^{2-1}c^{4-(-2)} = \mathbf{a^2bc^6}$$

When $a = 3$, $b = \frac{1}{8}$ and $c = 2$,

$$a^2bc^6 = (3)^2\left(\frac{1}{8}\right)(2)^6 = (9)\left(\frac{1}{8}\right)(64) = \mathbf{72}$$

Problem 4. Simplify $\dfrac{x^2y^3 + xy^2}{xy}$

$$\frac{x^2y^3 + xy^2}{xy} = \frac{x^2y^3}{xy} + \frac{xy^2}{xy}$$

$$= x^{2-1}y^{3-1} + x^{1-1}y^{2-1}$$

$$= \mathbf{xy^2 + y} \quad \text{or} \quad \mathbf{y(xy + 1)}$$

Problem 5. Simplify $\dfrac{(x^2\sqrt{y})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{\frac{1}{2}}}$

$$\frac{(x^2\sqrt{y})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{\frac{1}{2}}} = \frac{x^2y^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{2}{3}}}{x^{\frac{5}{2}}y^{\frac{3}{2}}}$$

$$= x^{2+\frac{1}{2}-\frac{5}{2}}y^{\frac{1}{2}+\frac{2}{3}-\frac{3}{2}}$$

$$= x^0y^{-\frac{1}{3}}$$

$$= \mathbf{y^{-\frac{1}{3}}} \quad \text{or} \quad \mathbf{\frac{1}{y^{\frac{1}{3}}}} \quad \text{or} \quad \mathbf{\frac{1}{\sqrt[3]{y}}}$$

Now try the following Practice Exercise

Practice Exercise 1 Basic algebraic operations and laws of indices (Answers on page 830)

1. Evaluate $2ab + 3bc - abc$ when $a = 2$, $b = -2$ and $c = 4$
2. Find the value of $5pq^2r^3$ when $p = \frac{2}{5}$, $q = -2$ and $r = -1$
3. From $4x - 3y + 2z$ subtract $x + 2y - 3z$.
4. Multiply $2a - 5b + c$ by $3a + b$
5. Simplify $(x^2y^3z)(x^3yz^2)$ and evaluate when $x = \frac{1}{2}$, $y = 2$ and $z = 3$
6. Evaluate $(a^{\frac{3}{2}}bc^{-3})(a^{\frac{1}{2}}b^{-\frac{1}{2}}c)$ when $a = 3$, $b = 4$ and $c = 2$
7. Simplify $\dfrac{a^2b + a^3b}{a^2b^2}$
8. Simplify $\dfrac{(a^3b^{\frac{1}{2}}c^{-\frac{1}{2}})(ab)^{\frac{1}{3}}}{(\sqrt{a^3}\sqrt{b}\,c)}$

(b) Brackets, factorisation and precedence

Problem 6. Simplify $a^2 - (2a - ab) - a(3b + a)$

$$a^2 - (2a - ab) - a(3b + a)$$

$$= a^2 - 2a + ab - 3ab - a^2$$

$$= \mathbf{-2a - 2ab} \quad \text{or} \quad \mathbf{-2a(1 + b)}$$

Problem 7. Remove the brackets and simplify the expression:

$$2a - [3\{2(4a - b) - 5(a + 2b)\} + 4a]$$

Removing the innermost brackets gives:

$$2a - [3\{8a - 2b - 5a - 10b\} + 4a]$$

Collecting together similar terms gives:

$$2a - [3\{3a - 12b\} + 4a]$$

Removing the 'curly' brackets gives:

$$2a - [9a - 36b + 4a]$$

Collecting together similar terms gives:

$$2a - [13a - 36b]$$

Removing the square brackets gives:

$$2a - 13a + 36b = \mathbf{-11a + 36b} \quad \text{or} \quad \mathbf{36b - 11a}$$

Problem 8. Factorise (a) $xy - 3xz$ (b) $4a^2 + 16ab^3$ (c) $3a^2b - 6ab^2 + 15ab$

(a) $xy - 3xz = \mathbf{x(y - 3z)}$

(b) $4a^2 + 16ab^3 = \mathbf{4a(a + 4b^3)}$

(c) $3a^2b - 6ab^2 + 15ab = \mathbf{3ab(a - 2b + 5)}$

Problem 9. Simplify $3c + 2c \times 4c + c \div 5c - 8c$

The order of precedence is division, multiplication, addition, and subtraction (sometimes remembered by BODMAS). Hence

$$3c + 2c \times 4c + c \div 5c - 8c$$
$$= 3c + 2c \times 4c + \left(\frac{c}{5c}\right) - 8c$$
$$= 3c + 8c^2 + \frac{1}{5} - 8c$$
$$= \mathbf{8c^2 - 5c + \frac{1}{5}} \quad \text{or} \quad \mathbf{c(8c - 5) + \frac{1}{5}}$$

Problem 10. Simplify $(2a - 3) \div 4a + 5 \times 6 - 3a$

$$(2a - 3) \div 4a + 5 \times 6 - 3a$$
$$= \frac{2a - 3}{4a} + 5 \times 6 - 3a$$
$$= \frac{2a - 3}{4a} + 30 - 3a$$
$$= \frac{2a}{4a} - \frac{3}{4a} + 30 - 3a$$
$$= \frac{1}{2} - \frac{3}{4a} + 30 - 3a = \mathbf{30\frac{1}{2} - \frac{3}{4a} - 3a}$$

Now try the following Practice Exercise

Practice Exercise 2 Brackets, factorisation and precedence (Answers on page 830)

1. Simplify $2(p + 3q - r) - 4(r - q + 2p) + p$
2. Expand and simplify $(x + y)(x - 2y)$
3. Remove the brackets and simplify:
   $24p - [2\{3(5p - q) - 2(p + 2q)\} + 3q]$
4. Factorise $21a^2b^2 - 28ab$
5. Factorise $2xy^2 + 6x^2y + 8x^3y$
6. Simplify $2y + 4 \div 6y + 3 \times 4 - 5y$
7. Simplify $3 \div y + 2 \div y - 1$
8. Simplify $a^2 - 3ab \times 2a \div 6b + ab$

1.3 Revision of equations

(a) Simple equations

Problem 11. Solve $4 - 3x = 2x - 11$

Since $4 - 3x = 2x - 11$ then $4 + 11 = 2x + 3x$

i.e. $15 = 5x$ from which, $\boldsymbol{x} = \frac{15}{5} = \mathbf{3}$

Problem 12. Solve

$$4(2a - 3) - 2(a - 4) = 3(a - 3) - 1$$

Removing the brackets gives:

$$8a - 12 - 2a + 8 = 3a - 9 - 1$$

Rearranging gives:

$$8a - 2a - 3a = -9 - 1 + 12 - 8$$

i.e. $3a = -6$

and $\boldsymbol{a} = \dfrac{-6}{3} = \mathbf{-2}$

Problem 13. Solve $\dfrac{3}{x-2} = \dfrac{4}{3x+4}$

By 'cross-multiplying': $3(3x+4) = 4(x-2)$

Removing brackets gives: $9x + 12 = 4x - 8$

Rearranging gives: $9x - 4x = -8 - 12$

i.e. $5x = -20$

and $\boldsymbol{x} = \dfrac{-20}{5} = \mathbf{-4}$

Problem 14. Solve $\left(\dfrac{\sqrt{t}+3}{\sqrt{t}}\right) = 2$

$$\sqrt{t}\left(\frac{\sqrt{t}+3}{\sqrt{t}}\right) = 2\sqrt{t}$$

i.e. $\sqrt{t} + 3 = 2\sqrt{t}$

and $3 = 2\sqrt{t} - \sqrt{t}$

i.e. $3 = \sqrt{t}$

and $\mathbf{9 = \boldsymbol{t}}$

(c) Transposition of formulae

Problem 15. Transpose the formula $v = u + \dfrac{ft}{m}$ to make f the subject.

$u + \dfrac{ft}{m} = v$ from which, $\dfrac{ft}{m} = v - u$

and $m\left(\dfrac{ft}{m}\right) = m(v-u)$

i.e. $ft = m(v-u)$

and $\boldsymbol{f = \dfrac{m}{t}(v-u)}$

Problem 16. The impedance of an a.c. circuit is given by $Z = \sqrt{R^2 + X^2}$. Make the reactance X the subject.

$\sqrt{R^2 + X^2} = Z$ and squaring both sides gives

$R^2 + X^2 = Z^2$, from which,

$X^2 = Z^2 - R^2$ and **reactance** $\boldsymbol{X = \sqrt{Z^2 - R^2}}$

Problem 17. Given that $\dfrac{D}{d} = \sqrt{\left(\dfrac{f+p}{f-p}\right)}$ express p in terms of D, d and f.

Rearranging gives: $\sqrt{\left(\dfrac{f+p}{f-p}\right)} = \dfrac{D}{d}$

Squaring both sides gives: $\dfrac{f+p}{f-p} = \dfrac{D^2}{d^2}$

'Cross-multiplying' gives:

$$d^2(f+p) = D^2(f-p)$$

Removing brackets gives:

$$d^2 f + d^2 p = D^2 f - D^2 p$$

Rearranging gives: $d^2 p + D^2 p = D^2 f - d^2 f$

Factorising gives: $p(d^2 + D^2) = f(D^2 - d^2)$

and $\boldsymbol{p = \dfrac{f(D^2 - d^2)}{(d^2 + D^2)}}$

Now try the following Practice Exercise

Practice Exercise 3 Simple equations and transposition of formulae (Answers on page 830)

In problems 1 to 4 solve the equations

1. $3x - 2 - 5x = 2x - 4$
2. $8 + 4(x-1) - 5(x-3) = 2(5 - 2x)$
3. $\dfrac{1}{3a-2} + \dfrac{1}{5a+3} = 0$
4. $\dfrac{3\sqrt{t}}{1-\sqrt{t}} = -6$
5. Transpose $y = \dfrac{3(F-f)}{L}$ for f

6. Make l the subject of $t = 2\pi\sqrt{\frac{l}{g}}$
7. Transpose $m = \dfrac{\mu L}{L + rCR}$ for L
8. Make r the subject of the formula $$\frac{x}{y} = \frac{1 + r^2}{1 - r^2}$$

(d) Simultaneous equations

Problem 18. Solve the simultaneous equations:

$$7x - 2y = 26 \quad (1)$$

$$6x + 5y = 29 \quad (2)$$

$5 \times$ equation (1) gives:

$$35x - 10y = 130 \quad (3)$$

$2 \times$ equation (2) gives:

$$12x + 10y = 58 \quad (4)$$

Equation (3) + equation (4) gives:

$$47x + 0 = 188$$

from which, $\boldsymbol{x} = \dfrac{188}{47} = \mathbf{4}$

Substituting $x = 4$ in equation (1) gives:

$$28 - 2y = 26$$

from which, $28 - 26 = 2y$ and $\boldsymbol{y} = \mathbf{1}$

Problem 19. Solve

$$\frac{x}{8} + \frac{5}{2} = y \quad (1)$$

$$11 + \frac{y}{3} = 3x \quad (2)$$

$8 \times$ equation (1) gives: $x + 20 = 8y$ (3)

$3 \times$ equation (2) gives: $33 + y = 9x$ (4)

i.e. $x - 8y = -20$ (5)

and $9x - y = 33$ (6)

$8 \times$ equation (6) gives: $72x - 8y = 264$ (7)

Equation (7) − equation (5) gives:

$$71x = 284$$

from which, $\boldsymbol{x} = \dfrac{284}{71} = \mathbf{4}$

Substituting $x = 4$ in equation (5) gives:

$$4 - 8y = -20$$

from which, $4 + 20 = 8y$ and $\boldsymbol{y} = \mathbf{3}$

(e) Quadratic equations

Problem 20. Solve the following equations by factorisation:
(a) $3x^2 - 11x - 4 = 0$
(b) $4x^2 + 8x + 3 = 0$

(a) The factors of $3x^2$ are $3x$ and x and these are placed in brackets thus:

$(3x \quad)(x \quad)$

The factors of -4 are $+1$ and -4 or -1 and $+4$, or -2 and $+2$. Remembering that the product of the two inner terms added to the product of the two outer terms must equal $-11x$, the only combination to give this is $+1$ and -4, i.e.,

$$3x^2 - 11x - 4 = (3x + 1)(x - 4)$$

Thus $(3x + 1)(x - 4) = 0$ hence

either $(3x + 1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{3}}$

or $(x - 4) = 0$ i.e. $\boldsymbol{x = 4}$

(b) $4x^2 + 8x + 3 = (2x + 3)(2x + 1)$

Thus $(2x + 3)(2x + 1) = 0$ hence

either $(2x + 3) = 0$ i.e. $\boldsymbol{x = -\frac{3}{2}}$

or $(2x + 1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{2}}$

Problem 21. The roots of a quadratic equation are $\frac{1}{3}$ and -2. Determine the equation in x.

If $\frac{1}{3}$ and -2 are the roots of a quadratic equation then,

$$\left(x - \frac{1}{3}\right)(x + 2) = 0$$

i.e. $x^2 + 2x - \frac{1}{3}x - \frac{2}{3} = 0$

i.e. $x^2 + \frac{5}{3}x - \frac{2}{3} = 0$

or $\mathbf{3x^2 + 5x - 2 = 0}$

Problem 22. Solve $4x^2 + 7x + 2 = 0$ giving the answer correct to 2 decimal places.

From the quadratic formula if $ax^2 + bx + c = 0$ then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence if $4x^2 + 7x + 2 = 0$

$$\text{then} \quad x = \frac{-7 \pm \sqrt{7^2 - 4(4)(2)}}{2(4)}$$

$$= \frac{-7 \pm \sqrt{17}}{8}$$

$$= \frac{-7 \pm 4.123}{8}$$

$$= \frac{-7 + 4.123}{8} \text{ or } \frac{-7 - 4.123}{8}$$

i.e. $\boldsymbol{x = -0.36}$ or $\boldsymbol{-1.39}$

Now try the following Practice Exercise

Practice Exercise 4 Simultaneous and quadratic equations (Answers on page 830)

In problems 1 to 3, solve the simultaneous equations

1. $8x - 3y = 51$

   $3x + 4y = 14$

2. $5a = 1 - 3b$

   $2b + a + 4 = 0$

3. $\frac{x}{5} + \frac{2y}{3} = \frac{49}{15}$

   $\frac{3x}{7} - \frac{y}{2} + \frac{5}{7} = 0$

4. Solve the following quadratic equations by factorisation:

   (a) $x^2 + 4x - 32 = 0$

   (b) $8x^2 + 2x - 15 = 0$

5. Determine the quadratic equation in x whose roots are 2 and -5

6. Solve the following quadratic equations, correct to 3 decimal places:

   (a) $2x^2 + 5x - 4 = 0$

   (b) $4t^2 - 11t + 3 = 0$

1.4 Polynomial division

Before looking at long division in algebra let us revise long division with numbers (we may have forgotten, since calculators do the job for us!).

For example, $\frac{208}{16}$ is achieved as follows:

$$\begin{array}{r}
13 \\
16 \overline{)\,208} \\
\underline{16} \\
48 \\
\underline{48} \\
\cdot\,\cdot \\
\end{array}$$

(1) 16 divided into 2 won't go

(2) 16 divided into 20 goes 1

(3) Put 1 above the zero

(4) Multiply 16 by 1 giving 16

(5) Subtract 16 from 20 giving 4

(6) Bring down the 8

(7) 16 divided into 48 goes 3 times

(8) Put the 3 above the 8

(9) $3 \times 16 = 48$

(10) $48 - 48 = 0$

Hence $\frac{208}{16} = \mathbf{13}$ exactly

Similarly, $\frac{172}{15}$ is laid out as follows:

$$\begin{array}{r}
11 \\
15 \overline{)\,172} \\
\underline{15} \\
22 \\
\underline{15} \\
7 \\
\end{array}$$

Hence $\frac{172}{15} = 11$ remainder 7 or $11 + \frac{7}{15} = \mathbf{11\frac{7}{15}}$

Below are some examples of division in algebra, which in some respects is similar to long division with numbers.

(Note that a **polynomial** is an expression of the form

$$f(x) = a + bx + cx^2 + dx^3 + \cdots$$

and **polynomial division** is sometimes required when resolving into partial fractions – see Chapter 2.)

Problem 23. Divide $2x^2 + x - 3$ by $x - 1$

$2x^2 + x - 3$ is called the **dividend** and $x - 1$ the **divisor**. The usual layout is shown below with the dividend and divisor both arranged in descending powers of the symbols.

$$\begin{array}{r|l} & 2x + 3 \\ \hline x - 1 & 2x^2 + x - 3 \\ & 2x^2 - 2x \\ \hline & \quad\; 3x - 3 \\ & \quad\; 3x - 3 \\ \hline & \quad\;\; \cdot \quad \cdot \end{array}$$

Dividing the first term of the dividend by the first term of the divisor, i.e. $\frac{2x^2}{x}$ gives $2x$, which is put above the first term of the dividend as shown. The divisor is then multiplied by $2x$, i.e. $2x(x-1) = 2x^2 - 2x$, which is placed under the dividend as shown. Subtracting gives $3x - 3$. The process is then repeated, i.e. the first term of the divisor, x, is divided into $3x$, giving $+3$, which is placed above the dividend as shown. Then $3(x-1) = 3x - 3$, which is placed under the $3x - 3$. The remainder, on subtraction, is zero, which completes the process.

Thus $\mathbf{(2x^2 + x - 3) \div (x - 1) = (2x + 3)}$

[A check can be made on this answer by multiplying $(2x + 3)$ by $(x - 1)$ which equals $2x^2 + x - 3$.]

Problem 24. Divide $3x^3 + x^2 + 3x + 5$ by $x + 1$

$$\begin{array}{r|l} & (1)\;\;(4)\;\;(7) \\ & 3x^2 - 2x + 5 \\ \hline x + 1 & 3x^3 + x^2 + 3x + 5 \\ & 3x^3 + 3x^2 \\ \hline & \quad -2x^2 + 3x + 5 \\ & \quad -2x^2 - 2x \\ \hline & \qquad\qquad 5x + 5 \\ & \qquad\qquad 5x + 5 \\ \hline & \qquad\qquad\; \cdot \quad \cdot \end{array}$$

(1) x into $3x^3$ goes $3x^2$. Put $3x^2$ above $3x^3$

(2) $3x^2(x + 1) = 3x^3 + 3x^2$

(3) Subtract

(4) x into $-2x^2$ goes $-2x$. Put $-2x$ above the dividend

(5) $-2x(x + 1) = -2x^2 - 2x$

(6) Subtract

(7) x into $5x$ goes 5. Put 5 above the dividend

(8) $5(x + 1) = 5x + 5$

(9) Subtract

Thus $\frac{3x^3 + x^2 + 3x + 5}{x + 1} = \mathbf{3x^2 - 2x + 5}$

Problem 25. Simplify $\frac{x^3 + y^3}{x + y}$

$$\begin{array}{r|l} & (1)\;\;(4)\;\;(7) \\ & x^2 - xy + y^2 \\ \hline x + y & x^3 + 0 + 0 + y^3 \\ & x^3 + x^2y \\ \hline & \quad -x^2y \qquad + y^3 \\ & \quad -x^2y - xy^2 \\ \hline & \qquad\qquad xy^2 + y^3 \\ & \qquad\qquad xy^2 + y^3 \\ \hline & \qquad\qquad\; \cdot \quad \cdot \end{array}$$

(1) x into x^3 goes x^2. Put x^2 above x^3 of dividend

(2) $x^2(x + y) = x^3 + x^2y$

(3) Subtract

(4) x into $-x^2y$ goes $-xy$. Put $-xy$ above dividend

(5) $-xy(x + y) = -x^2y - xy^2$

(6) Subtract

(7) x into xy^2 goes y^2. Put y^2 above dividend

(8) $y^2(x + y) = xy^2 + y^3$

(9) Subtract

Thus

$$\frac{x^3 + y^3}{x + y} = \mathbf{x^2 - xy + y^2}$$